Path Planning on Grids: The Effect of Vertex Placement on Path Length[*]

James Bailey   Craig Tovey
School of ISYE
Georgia Institute of Technology
Atlanta, Georgia, USA
{james.bailey,ctovey}@gatech.edu

Tansel Uras   Sven Koenig
Department of Computer Science
University of Southern California
Los Angeles, California, USA
{turas,skoenig}@usc.edu

Alex Nash
Northrop Grumman
Information Systems
Redondo Beach, California, USA
alex.nash@ngc.com

Abstract

Video-game designers often tessellate continuous 2-dimensional terrain into a grid of blocked and unblocked square cells. The three main ways to calculate short paths on such a grid are to determine truly shortest paths, shortest vertex paths and shortest grid paths, listed here in decreasing order of computation time and increasing order of resulting path length. We show that, for both vertex and grid paths on both 4-neighbor and 8-neighbor grids, placing vertices at cell corners rather than at cell centers tends to result in shorter paths. We quantify the advantage of cell corners over cell centers theoretically with tight worst-case bounds on the ratios of path lengths, and empirically on a large set of benchmark test cases. We also quantify the advantage of 8-neighbor grids over 4-neighbor grids.

Introduction

Video-game designers often tessellate continuous 2-dimensional terrain to form square grids (Tozour 2004). Examples include Dawn of War (1 and 2) and Company of Heroes (Champandard 2010). We assume that each cell is either completely blocked (grey) or unblocked (white) and that the traversal costs of unblocked cells are uniform, which is reasonable for many video games. One ought to study the properties of path planning on such grids, so that video-game designers can make informed choices. For example, if the size of game characters is small relative to the cell size, then it is reasonable to place the vertices of grid graphs at either the cell centers or cell corners. In this paper, we therefore study the effect of two design decisions, namely vertex placement (at either cell centers or cell corners) and grid connectivity (either 4 or 8 neighbors) on one particular metric of interest, namely the lengths of the resulting paths.

The three main ways to calculate short paths on grids are to determine truly shortest paths (TSP) in the continuous terrain (Lozano-Pérez and Wesley 1979; Mitchell and Papadimitriou 1991; Harabor and Grastien 2013), shortest vertex paths (SVP, sometimes also called shortest any-angle paths since many any-angle search algorithms, such as Block A* (Yap et al. 2011), Theta* (Daniel et al. 2010) and A* with Post-Smoothed Paths (Millington and Funge 2009), strive to find such paths although they are not guaranteed to succeed) and shortest grid paths (SGP), listed here in decreasing order of computation time and increasing order of resulting path length (Uras and Koenig 2015). SVPs or SGPs, respectively, are determined on the grid graphs that are constructed from the terrain by placing vertices at either all cell centers or all cell corners and then connecting any two vertices or any two neighboring vertices, respectively, if the line segment between them is unblocked. See (Nash and Koenig 2013) for more information about TSPs, SVPs and SGPs and about search algorithms based on A* (Hart, Nilsson, and Raphael 1968) that determine them. For a given vertex placement, SGPs cannot (by definition) be shorter than the corresponding SVPs, and SVPs cannot be shorter than the corresponding TSPs. However, it has not previously been known how large the ratios among these path lengths can be. Our theoretical results are a complete set of matching (hence tight) upper and lower bounds with respect to the worst-case ratios of the three kinds of paths. Our main contribution is the analysis of the design decision to place vertices at cell corners for both grid connectivities, especially since several of the resulting bounds turn out to be non-trivial.

Table 1 summarizes our theoretical results. All worst-case ratios are tight (or asymptotically tight), that is, there exist path-planning problems for which the worst-case ratios have the given values (or are within any $\epsilon > 0$ of the given values) but no path-planning problems for which the worst-case ratios are larger. Values that are only asymptotically tight are indicated by an asterisk ($*$). The table shows that the vertex placement can indeed dramatically affect the worst-case ratio of the lengths of an SGP and the corresponding TSP and the worst-case ratio of the lengths of an SVP and the corresponding TSP. For example, we prove worst-case ratios $\frac{SGP}{TSP}$ of ≈ 1.76 (for vertices at cell centers) versus ≈ 1.41 (for vertices at cell corners) on 4-neighbor grids and ≈ 1.24 versus ≈ 1.08 on 8-neighbor grids. We also prove that the

worst-case ratio between the lengths of an SGP and the corresponding SVP remains, perhaps surprisingly, unaffected by the vertex placement for both grid connectivities.

Finally, we compare the influence of vertex placement and grid connectivity empirically on a large set of benchmark cases and verify that placing vertices at cell corners rather than cell centers indeed leads to shorter path lengths. These results provide valuable information to video-game designers when it comes to making design decisions about vertex placement and grid connectivity, even if they use search algorithms that only strive to find TSPs, SVPs or SGPs but are not guaranteed to succeed.

Overview and Related Work

Terrain is modeled as a 4-fold symmetric tessellation of a finite portion of the plane into unit side length square cells, each of which is either completely blocked or unblocked. The standard grid graph corresponding to the terrain has vertices at either all unblocked cell centers (resulting in a center grid graph) or all unblocked cell corners (resulting in a corner grid graph). The edges of the graph connect every vertex to each of its 4 or 8 neighboring vertices in the main compass directions, except that edges that would have to squeeze between blocked cells are not permitted. Consequently, we distinguish four scenarios, namely 4-corner, 4-center, 8-corner and 8-center grid graphs. Cardinal direction edges have length 1, and diagonal edges have length $\sqrt{2}$.

We are interested in computing the worst-case ratio $\frac{SGP(s,t)}{TSP(s,t)}$ (short: $\frac{SGP}{TSP}$) of the lengths of an SGP and the corresponding TSP (on the same grid and with the same start and goal vertices), the worst-case ratio $\frac{SVP(s,t)}{TSP(s,t)}$ (short: $\frac{SVP}{TSP}$) of the lengths of an SVP and the corresponding TSP and the worst-case ratio $\frac{SGP(s,t)}{SVP(s,t)}$ (short: $\frac{SGP}{SVP}$) of the lengths of an SGP and the corresponding SVP, maximized over all grid sizes, start vertices s, goal vertices t and arrangements of blocked cells, for the four scenarios. Some relationships are known or obvious:

- If there exists a TSP, there always exists a piecewise linear TSP that turns only at cell corners (Lee 1978; Lozano-Pérez and Wesley 1979). Thus, TSPs and SVPs are equally long if the vertices are placed at cell corners. Consequently, $b = j$, $d = l$ and $f = h = 1$ in Table 1.
- SVPs and TSPs do not depend on the grid connectivity. Consequently, $e = g$ and $f = h$ in Table 1.

These relationships imply that only the worst-case ratios a, c, g, i, j, k and l need to be determined in Table 1. Previous research has studied some of these worst-case ratios under different assumptions. In particular, worst-case ratios b and d were previously known to be $\sqrt{2}$ and $\sqrt{4 - 2\sqrt{2}}$, respectively, if the paths are allowed to pass through diagonally-touching blocked cells, which was first analyzed under the assumption that all cells are unblocked (Ferguson and Stentz 2006) and later generalized to the case where cells can be blocked or unblocked (Nash 2012). Our theoretical results show that these worst-case ratios do not change even under our assumption that paths are not allowed to pass through

diagonally-touching blocked cells, which is the more realistic assumption for video games since game characters have non-zero sizes. Also, some worst-case ratios become infinite if paths were allowed to pass though diagonally-touching blocked cells. For example, Figure 1 shows a path-planning problem on a 4-neighbor grid with vertices at cell centers. The start vertex is s, and the goal vertex is t. The length of the TSP would be finite (as shown in Figure 1) while the length of the SGP is infinite (that is, there is no such path). The worst-case ratio a would be infinite instead of 1.76. More importantly, it is quite common to place vertices at cell centers, and this case has not been studied before to the best of our knowledge, which is why we analyze whether vertex placement matters for the worst-case ratios.

Figure 1: If movement between diagonally-touching blocked cells were permitted, $TSP(s,t) = \sqrt{2}$ but t could not be reached from s by a grid path on the 4-center grid graph.

Points and Paths

Points are pairs of coordinates. Their first coordinate is the x-coordinate, which increases to the right. Their second coordinate is the y-coordinate, which increases to the top. We now define different kinds of paths.

Definition 1. *An* s-t *path is a path in the plane from point s to point t such that (i) it is contained in the union of unblocked cells (where the region of a cell includes its perimeter); and (ii) may not pass through diagonally-touching blocked cells (that is, if it passes through a cell corner c contained in two blocked cells that intersect only at c, it exits c to the same unblocked cell from which it entered).*

Definition 2. $TSP(s,t)$, *the length of a truly shortest path (TSP), is the (Euclidean) length of a shortest* s-t *path.*

For all instances for which no TSPs exist, the lengths of the TSPs, SVPs and SGPs are all infinity, which is consistent with Table 1. We therefore consider from now on only instances for which TSPs exist.

For center grid graphs $G = (V, E, U)$, the vertex set V contains all cell centers of the unblocked cells. For corner grid graphs, V contains all cell corners of the unblocked cells. A point that is the cell corner of more than one unblocked cell is represented as a single element of V. For 4-neighbor grids, the edge set E is obtained by connecting vertex v in V with coordinates (x, y) to each vertex w at coordinates $(x \pm 1, y)$ and $(x, y \pm 1)$ if and only if the line segment $\overline{vw}$ between them is a *v-w* path. For 8-neighbor grids, E contains additional edges since v is also connected to each vertex w at coordinates $(x \pm 1, y \pm 1)$ if and only if the line segment $\overline{vw}$ between them is a *v-w* path. The description of G also includes the list of unblocked cells U.

Worst-Case Ratio	4-Neighbor Grids		8-Neighbor Grids	
	Vertices at Centers (4-Center Grid Graphs)	Vertices at Corners (4-Corner Grid Graphs)	Vertices at Centers (8-Center Grid Graphs)	Vertices at Corners (8-Corner Grid Graphs)
shortest grid path/truly shortest path ($\frac{SGP}{TSP}$)	$a = 6/(2+\sqrt{2}) \approx 1.7574$	$b = \sqrt{2} \approx 1.4142$	$c = 3\sqrt{2} - 3 \approx 1.2426$	$d = \sqrt{4-2\sqrt{2}} \approx 1.0824^*$
shortest vertex path/truly shortest path ($\frac{SVP}{TSP}$)	$e = 3\sqrt{2} - 3 \approx 1.2426$	$f = 1$	$g = 3\sqrt{2} - 3 \approx 1.2426$	$h = 1$
shortest grid path/shortest vertex path ($\frac{SGP}{SVP}$)	$i = \sqrt{2} \approx 1.4142$	$j = \sqrt{2} \approx 1.4142$	$k = \sqrt{4-2\sqrt{2}} \approx 1.0824^*$	$l = \sqrt{4-2\sqrt{2}} \approx 1.0824^*$

Table 1: Theoretical results.

Definition 3. $SGP(s,t)$, *the length of a shortest* s-t *grid path (SGP), is the (Euclidean) length of a shortest* s-t *path in grid graph G.*

We want to compare the lengths of *s-t* SGPs with those of the corresponding *s-t* TSPs and *s-t* SVPs. The latter are defined in terms of *s-t* paths on a different graph:

Definition 4. *Let $G = (V, E, U)$ be a grid graph. The* vertex graph $G' = (V, E', U)$ *is defined by edge set E' consisting of all $(v, w) : v \in V, w \in V$ such that the line segment between them is a* v-w *path. $SVP(s,t)$, the length of a shortest* s-t *vertex path (SVP), is the (Euclidean) length of a shortest* s-t *path in graph G'.*

For brevity, we omit most proofs of the correctness of the following definitions and properties. All omitted proofs consist of an enumeration of the possible sets of unblocked cells incident on a vertex, or the treatment of paths parallel to an axis separate from paths not parallel to an axis.

For any vertices s and t, if there exist multiple *s-t* TSPs, we may select one of those paths and retain in U only a smallest set of cells that permit it to be an *s-t* path. The TSP is then unique. The additional blocked cells do not change $TSP(s,t)$ and cannot decrease $SVP(s,t)$ or $SGP(s,t)$. Thus, we assume that the TSP is unique when proving upper bounds on the worst-case ratios $\frac{SVP}{TSP}$ and $\frac{SGP}{TSP}$.

Definition 5. *Let $x_1, x_2, ..., x_n$ be the points where the* s-t *TSP turns. For all $1 \leq i \leq n$, x_i must be on the cell corner of exactly one blocked cell, denoted F_i. There exists a unique unblocked cell B_i (the one diagonally-touching F_i) such that $F_i \cap B_i = x_i$. Define c_{B_i} to be the cell center of B_i. Define $x_0 = s$ and $x_{n+1} = t$.*

We now introduce the key idea that enables us to prove upper bounds on the worst-case ratio $\frac{SGP}{TSP}$. We define a restricted class of grid paths, called *restricted grid paths (RGP)*, that stay "close" to the TSP. Since any RGP is a grid path, an SGP is at least as short as a shortest RGP. Hence any upper bound on the length of the latter is an upper bound on the length of the former. Informally, an RGP traverses only cells that the TSP traverses, traverses all vertices that the TSP traverses and always turns in a cell where the TSP turns. Formally:

Definition 6. *For any vertex v, let $P(v)$ be the set of cells that contain v. Similarly, for any line segment $\overline{vw}$, let $P(v, w)$ be the set of cells that intersect $\overline{vw}$. For 4-corner, 8-corner and 8-center grid graphs, let $y_i = x_i$ for all i, where x_i is as in Definition 5. For 4-center grid graphs, let $y_0 = x_0 = s, y_{n+1} = x_{n+1} = t$ and $y_i = c_{B_i}$ for $1 \leq i \leq n$. An* s-t *restricted grid path (RGP) is an* s-t *path in G that satisfies the following conditions:*

- *If the TSP contains $v \in V$, the RGP must also contain v.*

- *The RGP divides into $n + 1$ y_i-y_{i+1} paths, respectively contained in $P(x_i, x_{i+1})$.*

$SRGP(s,t)$ *is the (Euclidean) length of a shortest* s-t *restricted grid path (SRGP).*

The SRGP and the SGP can be very different in both shape and length as shown in Figure 2 for a 4-neighbor grid with vertices at cell centers. The dashed path is the TSP, the dotted path is the SGP, and the solid path is the SRGP. Points y_1 and y_2 are those required in the definition of the RGP.
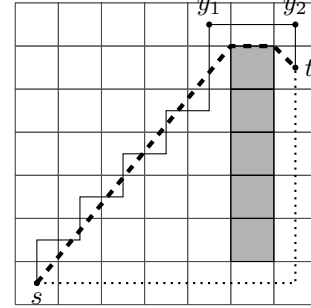

Figure 2: The SGP and SRGP have different shapes and lengths for this 4-center grid graph.

We omit the straightforward proof that an SRGP always exists and make repeated use of the following simple formulas for $SRGP(s,t)$ that demonstrate that, under the condition given, an SGP can stay close to the TSP, resulting in the SRGP and SGP being equally long. An example for the construction that proves Lemma 1 for 8-center grid graphs (Lemma 2 for 4-corner grid graphs) is shown in Figure 3 (Figure 4).

Lemma 1. *For any 8-center (8-corner) grid graph G, let s and t be points in the interior of cells A and B, respectively, such that $\overline{st}$ is an s-t path. Let v_A and v_B be the cell centers (lower left and upper right corners, respectively) of A and B, respectively. Let $v_B - v_A = (p, q)$. If $p \geq q \geq 0$, then $SRGP(v_A, v_B) = \sqrt{2}q + p - q$.*

Proof. The lower bound follows since this is the length of an SGP if there are no blocked cells. By translation assume that $v_A = (0, 0)$ and $v_B = (p, q)$ where $p \geq q \geq 0$. Let $b(i, j)$ be the cell with cell center (upper right corner) at (i, j) where $i, j \in \mathbb{Z}$. For all $j \in \{1, ..., q\}$, let $f(j)$ be the smallest integer such that $\overline{st}$ enters the interior of $b(f(j), j)$ and let $g(j)$ be the maximum of j and $f(j)$. Since $p \geq q \geq 0$, $\overline{st}$ enters the interior of $b(g(j), j)$ and must also enter the interior of $b(g(j) - 1, j - 1)$. The grid path $v_A = (0, 0) -$
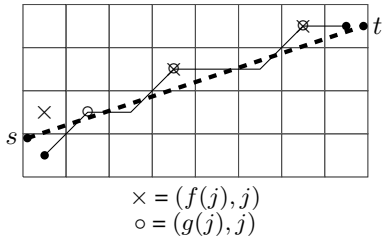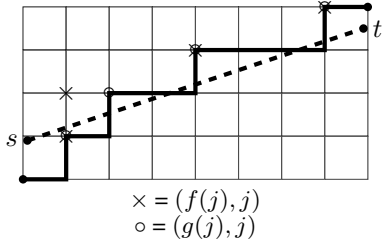
Figure 3: SRGP of Lemma 1 for 8-center grid graphs.

Figure 4: SRGP of Lemma 2 for 4-corner grid graphs.

$(g(1) - 1, 0) - (g(1), 1) - (g(2) - 1, 1) - (g(2), 2) - ... - (f(q), q) - (p, q) = v_B$ is a valid RGP with length $\sqrt{2}q + p - q$. $\square$

Lemma 2. *For any 4-center (4-corner) grid graph G, let s, t, A, B, v_A, v_B be as in Lemma 1. If $p \geq q \geq 0$, then $SRGP(v_A, v_B) = p + q$. In general, $SRGP(v_A, v_B) = |p| + |q|$.*

Proof. The general result is obtained by transforming the general case via rotation, reflection and translation to the special case $p \geq q \geq 0$. Therefore, assume that $p \geq q \geq 0$. The proof is then the same as for Lemma 1 with the following addition: For 4-corner grid graphs, each $(g(i) - 1, i - 1) - (g(i), i)$ in the path for 8-corner grid graphs is replaced by $(g(i) - 1, i - 1) - (g(i), i - 1) - (g(i), i)$, which is possible since $b(g(i), i)$ must be unblocked. For 4-center grid graphs, each $(g(i) - 1, i - 1) - (g(i), i)$ in the path for 8-center grid graphs is replaced by either $(g(i) - 1, i - 1) - (g(i) - 1, i) - (g(i), i)$ or $(g(i) - 1, i - 1) - (g(i), i - 1) - (g(i), i)$, which is possible since paths are not allowed to squeeze between two diagonally-touching blocked cells and thus $b(g(i) - 1, i)$ or $b(g(i), i - 1)$ must be unblocked. $\square$

Tight Bound for Worst-Case Ratio $\frac{SGP}{TSP}$ for 4-Center Grid Graphs

The proofs of two of the bounds reported here each require multiple case-by-case analyses. Rather than summarizing each, we give a fairly detailed proof for the worst-case ratio a in Table 1. Many of the principles employed here can also be used to prove the bound for worst-case ratio c in Table 1.

Theorem 3 (Worst-Case Ratio a in Table 1)**.** *For 4-center grid graphs, the worst-case ratio $\frac{SGP}{TSP}$ is $a = \frac{6}{2 + \sqrt{2}}$.*
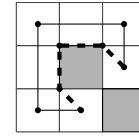

Figure 5: Worst-case ratio $\frac{SGP}{TSP}$ for 4-center grid graphs.

We first prove that, for 4-center grid graphs, the worst-case ratio $\frac{SRGP}{TSP}$ is $\frac{6}{2 + \sqrt{2}}$. Figure 5 establishes by example the lower bound $\alpha = \frac{6}{2 + \sqrt{2}}$ on the worst-case ratio $\frac{SRGP}{TSP}$. To prove that this lower bound is tight (that is, to prove an upper bound of α), we repeatedly utilize the lower bound. The lower bound is helpful because, throughout the proof, we consider any instance where $\frac{SRGP(s,t)}{TSP(s,t)} \geq \alpha$ for its start vertex s and goal vertex t since only such an instance can possibly achieve the worst-case ratio. We then try to move either s or t to another vertex to reduce $TSP(s, t)$ by at least $b > 0$ and $SRGP(s, t)$ by $a > 0$. This procedure is called α-*worsening* if and only if $\frac{a}{b} \leq \alpha$ because of the following property: Given that the ratio $\frac{SRGP(s,t)}{TSP(s,t)}$ of the instance was originally at least α, the procedure has made the ratio no smaller while decreasing $TSP(s, t)$ by $b > 0$. Thus, the instance at hand cannot be an instance that comes within ϵ of the worst-case ratio with minimal $TSP(s, t)$, called a *smallest instance*. Eventually, we will characterize the unique smallest instance (up to symmetry) as the one shown in Figure 5.

Consider any instance where $\frac{SRGP(s,t)}{TSP(s,t)} \geq \alpha$. We proceed by breaking the problem into cases based on the number of turns of the *s-t* TSP for the instance.

Zero Turns

Suppose that s and t are the respective cell centers of cells A and B such that $s - t = (p, q)$, and that the TSP is the line segment $\overline{st}$. Then, $TSP(s, t) = \sqrt{p^2 + q^2}$ and, by Lemma 2, $SRGP(s, t) = |p| + |q|$. Thus, $\frac{SRGP(s,t)}{TSP(s,t)} \leq \sqrt{2}$, a contradiction.

One or More Turns

Let x_i indicate the ith turn that the TSP makes, and let y_i be as described in Definition 6. The pair comprising an x_i-x_{i+1} path and y_i-y_{i+1} path is a *segment*. The pair comprising the s-x_1 and s-y_1 paths is the *head*. If x_n is the last vertex where the TSP makes a turn, then the pair comprising the x_n-t and y_n-t paths forms the *tail*.

Heads and Tails

Lemma 4. *Subject to rotation and reflection, every smallest instance for 4-center grid graphs must have the tail shown in Figure 6 if the TSP for the instance has at least one turn. Similarly, every smallest instance for 4-center grid graphs must have the head corresponding to the tail shown in Figure 6 under the same conditions.*

Proof. By reversing the roles of s and t it suffices to analyze only the tail. Through rotation and reflection, we may

Figure 6: Tail for smallest instance.

assume that the last turn, x_n, is the upper right cell corner of the blocked cell F_n that causes x_n and that there is a right turn at x_n. For any tail, we may move t in order to create the tail shown in Figure 6 since every tail requires the unblocked cells shown in Figure 6.

Suppose that $t - x_n = (p, q)$. Our assumptions imply that $p \geq \frac{1}{2}$ and $q \leq -\frac{1}{2}$, which means that $|p|, |q| \geq \frac{1}{2}$. $TSP(x_n, t) = \sqrt{p^2 + q^2}$. Consider the point s' reached by moving ϵ along $\overline{x_n t}$, and let A and B be the cells containing s' and t, respectively. Then, $SRGP(y_n, t) = |p| + |q|$ since both $SRGP(y_n, t) \leq SRGP(v_A, t) + 1 = SRGP(v_A, v_B) + 1 = (|p| - 1/2) + (|q| - 1/2) + 1 = |p| + |q|$ by Lemma 2 and $SRGP(y_n, t) \geq SGP(y_n, t) \geq |p| + |q|$. By moving t to the vertex shown in Figure 6, the TSP is shortened by $\sqrt{p^2 + q^2} - \frac{1}{\sqrt{2}}$ and the SRGP is shortened by $|p| + |q| - 1$. This is α-worsening if $|p| + |q| \geq 2$ since then $\frac{|p|+|q|-1}{\sqrt{p^2+q^2}-\frac{1}{\sqrt{2}}} \leq \sqrt{2} < \frac{6}{2+\sqrt{2}} = \alpha$. If $|p| + |q| = 1$, we have the tail shown in Figure 6. As a result, a smallest instance must use only the tail shown in Figure 6 and the head shown in Figure 7. $\square$

Segments When examining segment i, described by the x_i-x_{i+1} path and y_i-y_{i+1} path, we assume that there is a right turn at x_i and that x_i is located at the upper right cell corner of F_i. We begin by analyzing segment 1:

Lemma 5. *Segment 1, if it exists, in a smallest instance for 4-center grid graphs must be as shown in Figure 8.*
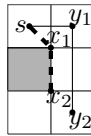

Figure 7: Prefix of path up to x_1.

Figure 8: Prefix of path up to x_2.

Proof. For all segments except the one shown in Figure 8, we give an α-worsening procedure. By Lemma 4, s, x_1 and y_1 are located as shown in Figure 7.
Case 1, $\overline{x_1 x_2}$ is not parallel to the y-axis: Suppose that $x_2 - x_1 = (p, q)$. We have $TSP(x_1, x_2) = \sqrt{p^2 + q^2}$ and

$SRGP(y_1, y_2) = |p| + |q|$ by Lemma 2. By moving s such that s, x_2 and y_2 form the head shown in Figure 6, we essentially remove segment 1 and thereby reduce the TSP by $\sqrt{p^2 + q^2}$ and the SRGP by $|p| + |q|$. This procedure is α-worsening, implying that these types of segments cannot occur in a smallest instance.
Case 2, Left turn at x_2 and $\overline{x_1 x_2}$ is parallel to the y-axis: Suppose that $x_1 - x_2 = (0, q)$ and $TSP(x_1, x_2) = |q|$. Since $\overline{x_1 x_2}$ is parallel to the y-axis, the y_1-y_2 SRGP is able to use only the columns containing s or y_1. The y_1-y_2 SRGP starts in the right column and may zig into the left column and zag into the right column in order to avoid blocked cells. Since y_2 is in the left column, the path must zig one more time than it zags. Let m be the number of times the y_1-y_2 SRGP zigs into the left column. Then $SRGP(y_1, y_2) = |q| + 2m$. If z_w and z_{w+1} denote two consecutive turns in the SRGP where z_w and z_{w+1} are in the same column, then $SRGP(z_w, z_{w+1}) \geq 2$ and thus $m \leq \lfloor \frac{|q|+1}{4} \rfloor$. Furthermore, $|q| \geq 3$ since otherwise we cannot travel in a straight line from x_1 to x_2. It now follows that it is α-worsening to move s such that s, x_2 and y_2 form the head shown in Figure 6, implying that these types of segments cannot occur in a smallest instance.
Case 3, Right turn at x_2 and $\overline{x_1 x_2}$ is parallel to the y-axis: Suppose that $x_1 - x_2 = (0, q)$ and $TSP(x_1, x_2) = |q|$. Again, the y_1-y_2 SRGP is only able to use the columns containing s or y_1. There now must be an equal number of zigs and zags. Using the same notation as before, $SRGP(y_1, y_2) = |q| + 2m + 1$. Again, $SRGP(z_w, z_{w+1}) \geq 2$ and thus $m \leq \lfloor \frac{|q|-1}{4} \rfloor$, where $|q| \geq 1$. It is α-worsening to move s such that s, x_2 and y_2 form the head shown in Figure 6 if $|q| \geq 2$, implying that these types of segments cannot occur in a smallest instance. If $|q| = 1$, the segment is as shown in Figure 8 and the lemma is proven. $\square$

Lemma 6. *Segment 2, if it exists, in a smallest instance for 4-center grid graphs must be geometrically similar to segment 1; in particular, $SRGP(y_2, y_3) = 2$ and $TSP(x_2, x_3) = 1$.*

The proof of this lemma follows in the same fashion as the proof of Lemma 5. The key difference is that here, in essence, we remove both segments 1 and 2. Then, for any possible segment 2, except the one shown in Figure 9, we are able to give an α-worsening procedure.

Lemma 7. *There is no segment 3 in a smallest instance for 4-center grid graphs.*

Proof. By Lemmata 4, 5 and 6, s, x_1, y_1, x_2, y_2, x_3 and y_3 are located as shown in Figure 9. The center right cell cannot be blocked since there could not be a right turn at x_3 otherwise. In addition, the lower right cell shown must be blocked since otherwise there is a shorter path from s to x_3.
Case 1, $\overline{x_3 x_4}$ is not parallel to the y-axis: Suppose that $x_3 - x_4 = (p, q)$. We have $TSP(x_3, x_4) = \sqrt{p^2 + q^2}$ and $SRGP(y_3, y_4) = |p| + |q|$ by Lemma 2. It is α-worsening to move s such that s, x_4 and y_4 form the head shown in Figure 6, implying that these types of segments cannot occur in a smallest instance.
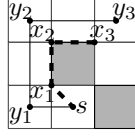
Figure 9: Prefix of path up to x_3.

Case 2, $\overline{x_3 x_4}$ is parallel to the y-axis: Because the lower right cell must be blocked (as argued above), this case cannot occur in a smallest instance. □

Construction of Smallest Instance

Under the justified assumption that the upper bound of the worst-case ratio $\frac{SRGP}{TSP}$ is at least $\alpha = \frac{6}{2+\sqrt{2}}$, we have proved that, if the TSP has zero turns, the worst-case ratio is at most $\sqrt{2} < \alpha$. – a contradiction. If the TSP has only one turn, by Lemma 4, the unique smallest instance (up to symmetry) can consist of only a head and tail as shown in Figure 10, which results in the TSP having zero turns – a contradiction. If the TSP has two turns, by Lemmata 4 and 5, the unique smallest instance (up to symmetry) is as shown in Figure 11, resulting in a worst-case ratio of $\frac{4}{1+\sqrt{2}} < \alpha$ – again a contradiction. If the TSP has three turns, then, by Lemmata 4, 5 and 6, the unique smallest instance (up to symmetry) is as shown in Figure 5, resulting in a worst-case ratio of $\frac{6}{2+\sqrt{2}} = \alpha$. By Lemma 7, the TSP cannot have four or more turns. This completes the proof of the upper bound $\frac{6}{2+\sqrt{2}}$ on the worst-case ratio $\frac{SRGP}{TSP}$. Note that Figures 10 and 11 illustrate the worst-case ratios if the TSP has fewer than two turns and two turns, respectively, but we don't prove this property since it is not required for the proof of Theorem 3 due to the assumption that the upper bound of the worst-case ratio is at least α.

Figure 5 establishes by example also the lower bound $\frac{6}{2+\sqrt{2}}$ on the worst-case ratio $\frac{SGP}{TSP}$. An upper bound on the worst-case ratio $\frac{SRGP}{TSP}$ is also an upper bound on the worst-case ratio $\frac{SGP}{TSP}$. Thus, the worst-case ratio is $\frac{SGP}{TSP}$ is $\frac{6}{2+\sqrt{2}}$ (as illustrated in Figure 5, which completes the proof of Theorem 3.

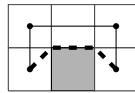Figure 10: Smallest instance for 4-center grid graphs if the TSP has fewer than two turns.

Figure 11: Smallest instance for 4-center grid graphs if the TSP has two turns.

Tight Bounds for Other Worst-Case Ratios

Theorem 8 (Worst-Case Ratios c and g in Table 1). *For 8-center grid graphs, the worst-case ratios $\frac{SGP}{TSP}$ and $\frac{SVP}{TSP}$ are* $c = g = \frac{3\sqrt{2}}{2+\sqrt{2}} \frac{2-\sqrt{2}}{2-\sqrt{2}} = 3\sqrt{2} - 3$.

Proof. The proof of the upper bound on the worst-case ratio $\frac{SGP}{TSP}$ follows from the same steps as the proof of Theorem 3. The upper bound on the worst-case ratio $\frac{SGP}{TSP}$ is also an upper bound on the worst-case ratio $\frac{SVP}{TSP}$ since SVPs are never longer than the corresponding SGPs. The lower bound for both worst-case ratios is shown in Figure 12 and coincides with the upper bound. □
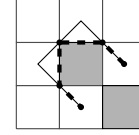

Figure 12: Worst-case ratios $\frac{SGP}{TSP}$ and $\frac{SVP}{TSP}$ for 8-center grid graphs.

For Theorems 9 and 10, we alter Definitions 5 and 6 and replace every instance of TSP with SVP. Specifically, we require in the proofs of Theorems 9 and 10 that the SRGP traverses only cells that the SVP traverses, traverses all vertices that the SVP traverses and always turns in a cell where the SVP turns.

Theorem 9 (Worst-Case Ratios i and j in Table 1). *For 4-center and 4-corner grid graphs, the worst-case ratio $\frac{SGP}{SVP}$ is $i = j = \sqrt{2}$.*

Proof. The *s-t* SVP of an instance that comes within ϵ of the worst-case ratio with minimal $SVP(s,t)$, called a smallest instance, cannot turn. If it turned at some vertex u, then the SVP and the SRGP would intersect at u. However, $\frac{SRGP(s,t)}{SVP(s,t)} = \frac{SRGP(s,u)+SRGP(u,t)}{SVP(s,u)+SVP(u,t)}$ implies $\max\{\frac{SRGP(s,u)}{SVP(s,u)}, \frac{SRGP(u,t)}{SVP(u,t)}\} \geq \frac{SRGP(s,t)}{SVP(s,t)}$, and the instance at hand cannot be smallest. Consequently, the *s-t* SVP of a smallest instance must be the line segment $\overline{st}$. Through rotation, reflection and translation, we may assume that $s = (0,0)$ and $t = (p,q)$ where $p \geq q \geq 0$. If $q = 0$, $\overline{st}$ is a valid grid path, yielding a worst-case ratio of 1. If $q \geq 1$, by Lemma 2, $\frac{SGP(s,t)}{SVP(s,t)} \leq \frac{SRGP(s,t)}{SVP(s,t)} = \frac{|p|+|q|}{\sqrt{p^2+q^2}} \leq \sqrt{2}$. An upper bound on the worst-case ratio $\frac{SRGP}{SVP}$ is also an upper bound on the worst-case ratio $\frac{SGP}{SVP}$. We achieve this upper bound if all cells are unblocked and $p = q$. □

Theorem 10 (Worst-Case Ratios k and l in Table 1). *For 8-center and 8-corner grid graphs, the worst-case ratio $\frac{SGP}{SVP}$ is asymptotically tight at $k = l = \sqrt{4 - 2\sqrt{2}}$.*

Proof. As in the proof of Theorem 9, assume that the *s-t* SVP is the line segment $\overline{st}$ with endpoints $s = (0,0)$ and $t = (p,q)$ where $p \geq q \geq 0$. If $q = 0$, $\overline{st}$ is a valid grid

	4-Neighbor Grids		8-Neighbor Grids	
	Game	Random	Game	Random
average $\frac{SGP(center)}{SGP(corner)}$	1.00427	1.03791	1.00193	1.01832
average $\frac{SGP(center)}{TSP(center)}$	1.2592	1.3106	1.0520	1.0662
average $\frac{SGP(corner)}{TSP(corner)}$	1.2536	1.2640	1.0496	1.0466
worst-case $\frac{SGP(center)}{TSP(center)}$	1.6569	1.7574	1.1716	1.2426
worst-case $\frac{SGP(corner)}{TSP(corner)}$	1.4142	1.4142	1.0824	1.0823
% $SGP(center) > SGP(corner)$	57.66%	67.34%	59.58%	74.43%
% $SGP(corner) > SGP(center)$	0.00%	0.00%	0.00%	2.71%

Table 2: Experimental results.

path, yielding a worst-case ratio of 1. If $q \geq 1$, by Lemma 1, $\frac{SGP(s,t)}{SVP(s,t)} \leq \frac{SRGP(s,t)}{SVP(s,t)} = \frac{\sqrt{2}q+p-q}{\sqrt{p^2+q^2}} = \frac{\sqrt{2}+(\beta-1)}{\sqrt{\beta^2+1}}$ where $\beta = p/q$. Setting the derivative to zero, we find that the ratio is maximized at $\beta = 1+\sqrt{2}$, with value $\frac{2\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \frac{\sqrt{4-2\sqrt{2}}}{\sqrt{4-2\sqrt{2}}} = \sqrt{4-2\sqrt{2}}$. An upper bound on the worst-case ratio $\frac{SRGP}{SVP}$ is also an upper bound on the worst-case ratio $\frac{SGP}{SVP}$. We get arbitrarily close to this upper bound if all cells are unblocked and $p = \lceil(\sqrt{2}+1)q\rceil$ as q grows large. $\square$

Experimental Results

We compare the lengths of paths on both random and game maps in Nathan Sturtevant's repository[1]. For center grid graphs, the start and goal vertices are placed at cell centers. For corner grid graphs, the start and goal vertices are shifted to the upper left cell corners. Table 2 reports the average ratio of the lengths of an SGP on a center grid graph and the SGP on the corresponding corner grid graph, the average and worst-case ratios of the lengths of the SGP and the corresponding TSP (separately for center and corner grid graphs), and the percentages of instances where the SGP on a center grid graph is strictly longer or shorter than the SGP on the corresponding corner grid graph.

All ratios in Table 2 show that, on average and in the worst-case, SGPs on center grid graphs are longer than SGPs on the corresponding corner grid graphs. In fact, they are on average 0.193%-3.791% longer, which makes sense since they cannot "hug" obstacles formed by blocked cells as closely. They are never strictly shorter, except for 2.71% of the instances for 8-neighbor random maps (which is due to the slightly different start and goal vertices on the corresponding center and corner grid graphs). However, they are strictly longer for 57.66%-74.43% of the instances. The random maps contain many small obstacles formed by blocked cells, while the game maps consist of large empty spaces (since the game maps model the terrain but not buildings or units), which might explain why the disadvantage of an SGP on a center grid graph over the SGP on the corresponding corner grid graph is typically larger for random maps than game maps. We expect the gap to decrease as the game maps are populated with building and units. Similarly, all ratios in Table 2 show that, on average and in the worst-case, SGPs on 4-neighbor grids are longer than SGPs on the corresponding 8-neighbor grids, which makes sense since they traverse

---

[1]http://movingai.com/benchmarks/

obstacle-free areas less effectively and cannot "hug" obstacles formed by blocked cells as closely. For all combinations of vertex placement and grid connectivity, the experimental worst-case ratios on random maps agree to 4 significant digits with the corresponding theoretical worst-case ratios in Table 1.

Conclusions

We have determined how the worst-case ratios between the lengths of a truly shortest path, a shortest vertex path and a shortest grid path vary. Our theoretical results (Table 1) quantify the advantage of placing vertices at cell corners rather than cell centers when finding shortest grid paths on 4- and 8-neighbor square grids. Our experimental results (Table 2) confirm these qualitative relationships. It is future research to extend our results to other tessellations, such as triangles, hexagons and texes (Björnsson et al. 2003).

References

Björnsson, Y.; Enzenberger, M.; Holte, R.; Schaeffer, J.; and Yap, P. 2003. Comparison of different grid abstractions for pathfinding on maps. In *Proceedings of the International Joint Conference on Artificial Intelligence*, 1511–1512.

Champandard, A. 2010. Personal Communication.

Daniel, K.; Nash, A.; Koenig, S.; and Felner, A. 2010. Theta*: Any-angle path planning on grids. *Journal of Artificial Intelligence Research* 39:533–579.

Ferguson, D., and Stentz, A. 2006. Using interpolation to improve path planning: The Field D* algorithm. *Journal of Field Robotics* 23(2):79–101.

Harabor, D., and Grastien, A. 2013. An optimal any-angle pathfinding algorithm. In *Proceedings of the International Conference on Automated Planning and Scheduling*, 308–311.

Hart, P.; Nilsson, N.; and Raphael, B. 1968. A formal basis for the heuristic determination of minimum cost paths. *IEEE Transactions on Systems Science and Cybernetics* 4(2):100–107.

Lee, D.-T. 1978. *Proximity and Reachability in the Plane*. Ph.D. Dissertation, University of Illinois at Urbana-Champaign.

Lozano-Pérez, T., and Wesley, M. 1979. An algorithm for planning collision-free paths among polyhedral obstacles. *Communications of the ACM* 22(10):560–570.

Millington, I., and Funge, J. 2009. *Artificial Intelligence for Games*. Morgan Kaufmann, second edition.

Mitchell, J., and Papadimitriou, C. 1991. The weighted region problem: Finding shortest paths through a weighted planar subdivision. *Journal of the ACM* 38(1):18–73.

Nash, A., and Koenig, S. 2013. Any-angle path planning. *Artificial Intelligence Magazine* 34(4):85–107.

Nash, A. 2012. *Any-Angle Path Planning*. Ph.D. Dissertation, University of Southern California. http://idm-lab.org/project-o.html.

Tozour, P. 2004. Search space representations. In Rabin, S., ed., *AI Game Programming Wisdom 2*. Charles River Media. 85–102.

Uras, T., and Koenig, S. 2015. An empirical comparison of any-angle path-planning algorithms. In *Proceedings of the Symposium on Combinatorial Search*.

Yap, P.; Burch, N.; Holte, R.; and Schaeffer, J. 2011. Any-angle path planning for computer games. In *Proceedings of the Conference on Artificial Intelligence and Interactive Digital Entertainment*.