Incremental ARA*: An Incremental Anytime Search Algorithm for Moving-Target Search

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Abstract

Moving-target search, where a hunter has to catch a moving target, is an important problem for video game developers. In our case, the hunter repeatedly moves towards the target and thus has to solve similar search problems repeatedly. We develop Incremental ARA* (I-ARA*) for this purpose, the first incremental anytime search algorithm for moving-target search in known terrain. We provide an error bound on the lengths of the paths found by I-ARA* and show experimentally in known four-neighbor gridworlds that I-ARA* can be used with smaller time limits between moves of the hunter than competing state-of-the-art moving-target search algorithms, namely repeated A*, G-FRA*, FRA*, and sometimes repeated ARA*. The hunter tends to make more moves with I-ARA* than repeated A*, G-FRA* or FRA*, which find shortest paths for the hunter, but fewer moves with I-ARA* than repeated ARA*, which finds suboptimal paths for the hunter like I-ARA*. Also, the error bounds on the lengths of the paths of the hunter tend to be smaller with I-ARA* than repeated ARA*.

Introduction

Moving-target search (Ishida and Korf 1991), where a hunter has to catch a moving target in known terrain, is an important problem for video game developers because video game characters often need to chase hostile characters or catch up with friendly characters. In our case, the hunter finds a path from its current location to the current location of the target and moves along it. If the target moves off the path, then the hunter repeats the process (Koenig, Likhachev, and Sun 2007). Thus, the hunter has to solve similar search problems repeatedly and has to do so fast to move smoothly. The video game company Bioware, for example, imposes a time limit of 1-3 ms per search (Bulitko et al. 2007a), which is not surprising for video games since their searches can use up to 70 percent of the available CPU time (Loh and Prakash 2009). Repeated A* with consistent h-values tends to run too slowly to stay within such stringent time limits between moves of the hunter but can be sped up in two different ways.¹

• First, incremental search algorithms (Koenig et al. 2004) modify A* to reduce the runtime per search over repeated A* by re-using information from previous searches to speed up the current search. G-FRA* (Sun, Yeo, and Koenig 2010), for example, is the currently fastest incremental moving-target search algorithm on known graphs, and FRA* optimizes it for known gridworlds (Sun, Yeo, and Koenig 2009). G-FRA*, FRA* and repeated A* with consistent h-values find shortest paths but the runtime of FRA* tends to be smaller than the one of G-FRA* and the runtime of G-FRA* tends to be smaller than the one of repeated A*.

• Second, weighted A* (Pohl 1970) performs an A* search with h-values obtained by multiplying consistent h-values with a constant user-provided weight $\epsilon > 1$. It trades off solution quality and efficiency: Weighted A* with weight $\epsilon$ finds an $\epsilon$-suboptimal path (a path that is at most a factor of $\epsilon$ longer than the shortest path) with a runtime that tends to be the smaller the larger the weight is. For weights larger than one, the runtime of weighted A* tends to be smaller than the one of A* with consistent h-values. However, it is difficult to identify the smallest weight that makes weighted A* run within a user-provided time limit. Anytime Repairing A* (ARA*) (Likhachev, Gordon, and Thrun 2003) avoids this problem by using incremental search to perform a series of weighted A* searches (each one called a repair iteration) with decreasing weights to find paths from the current location of the hunter to the current location of the target with smaller and smaller error bounds on their lengths until it finds a shortest path or the user-provided time limit has been reached.

In this paper, we develop Incremental ARA* (I-ARA*), the first incremental anytime search algorithm for moving-target search.
target search in known terrain. I-ARA* operates in the same way as repeated ARA*, except that it also uses incremental search (as used in G-FRA*) to speed up the first (and often the slowest) repair iteration of each search by reusing information from the previous search. I-ARA* operates in the same way as G-FRA*, except that it performs a series of (modified) weighted A* searches (that can result in a suboptimal path) instead of a single (modified) A* search (that results in a shortest path). We provide an error bound on the lengths of the paths found by I-ARA* (by showing that I-ARA* repair iterations with weight \( \epsilon \) find \( \epsilon \)-suboptimal paths) and show experimentally in known four-neighbor gridworlds that I-ARA* can be used with smaller time limits than repeated A*, G-FRA*, FRA*, and sometimes repeated ARA*. The hunter tends to make more moves with I-ARA* than repeated A*, G-FRA* or FRA*, which find shortest paths for the hunter, but fewer moves with I-ARA* than repeated ARA* with the same time limits and the same schedule for decreasing the weights, which finds suboptimal paths for the hunter like I-ARA*. Also, the error bounds on the lengths of the paths of the paths the hunter tend to be smaller with I-ARA* than repeated ARA* because I-ARA* is often able to decrease the weights more than repeated ARA* within the time limits.

**Notation**

Although I-ARA* can operate on arbitrary graphs, we describe its operation on four-neighbor gridworlds with blocked and unblocked cells for ease of description. \( S \) is the finite set of unblocked cells, \( s_{start} \in S \) is the current cell of the hunter and the start cell of each search, and \( s_{goal} \in S \) is the current cell of the target and the goal cell of each search. Neighbor(\( s \)) \( \subseteq S \) is the set of unblocked neighbors of cell \( s \in S \). parent(\( s \)) \( \in \) Neighbor(\( s \)) is the parent of cell \( s \in S \) in the search tree, where a search tree contains exactly \( s_{start} \) and all cells with parents, that is, whose parents are different from NULL. \( c(s, s') \) is the length of a shortest path from cell \( s \in S \) to cell \( s' \in S \) (measured in the number of moves needed to move along the path).

**Existing Research: Repeated ARA***

The hunter always knows the gridworld and the current cells of both itself and the target. The hunter and target alternate moves (but do not need to move). The hunter can always move from \( s_{start} \) to any of its neighboring unblocked cells. I-ARA* builds on Anytime Repairing A* (ARA*) (Likhachev, Gordon, and Thrun 2003). I-ARA* and repeated ARA* operate as follows until the hunter occupies the same cell as the target: They perform a series of weighted A* searches from \( s_{start} \) to \( s_{goal} \) with decreasing weights to find paths with smaller and smaller error bounds on their lengths until they find a shortest path or the user-provided time limit has been reached. (If they find that no path from \( s_{start} \) to \( s_{goal} \) exists, then they exit unsuccessfully since the hunter will never be able to occupy the same cell as the target.) Then, they move the hunter along the path. If the target moves off the path, then they repeat the process.

We explain repeated ARA* first. Repeated ARA* maintains several values for each cell \( s \in S \): (1) The \( g \)-value \( g(s) \) is the length of the shortest path from \( s_{start} \) to \( s \) found so far. Initially, it is infinity. (2) The \( v \)-value \( v(s) \) is the \( g \)-value at the time of the last expansion of cell \( s \). Initially, it is infinity. We call a cell \( s \) **locally consistent** if \( v(s) = g(s) \) and **locally inconsistent** otherwise. (3) The \( h \)-value \( b(s, s') \) is a user-provided approximation of the length of a shortest path from \( s \) to \( s' \). The \( h \)-values have to be consistent, that is, obey the triangle inequality (Pearl 1985). We use the Manhattan distances as consistent \( h \)-values throughout the paper. (4) The \( f \)-value \( f(s) = g(s) + \epsilon \times h(s, s_{goal}) \) is an approximation of the length of an \( \epsilon \)-suboptimal path from \( s_{start} \) via \( s \) to \( s_{goal} \). Where \( \epsilon \) is the weight of the current repair iteration.

We call each weighted A* search a repair iteration, and the series of weighted A* searches between two moves of the hunter an ARA* search. Repeated ARA* proceeds as follows, using the sets \( OPEN \), \( CLOSED \) and \( INCONS \): Repeated ARA* first sets \( g(s_{start}) \) to zero, \( OPEN \) to contain only \( s_{start} \), and \( CLOSED \) and \( INCONS \) to the empty set. Repeated ARA* then starts the first repair iteration by setting the weight of the current repair iteration \( \epsilon \) to a user-provided initial weight \( \epsilon_{init} \) (to find a path from \( s_{start} \) to \( s_{goal} \) with a large error bound on its length quickly) and then repeats the

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2 OPEN is the set of all locally inconsistent cells that have not yet been expanded in the current repair iteration, CLOSED is the set of all locally consistent cells that have been expanded in the current repair iteration, and INCONS is the set of all locally inconsistent cells that have been expanded in the current repair iteration.

![Figure 1: First ARA* Search](image-url)
The following procedure: Repeated ARA* deletes a cell $s$ with the smallest $f$-value from OPEN, inserts it into CLOSED and expands it by setting $v(s)$ to $g(s)$ and performing the following operations for each neighbor $s' \in \text{Neighbor}(s)$ with $g(s') > v(s) + c(s, s')$: Repeated ARA* sets $g(s')$ to $v(s) + c(s, s')$ and parent$(s')$ to $s$. If $s'$ is neither in OPEN, CLOSED nor INCONS, then repeated ARA* generates it by inserting it into OPEN. If $s'$ is in CLOSED, then repeated ARA* re-generates it by moving it to INCONS. Repeated ARA* terminates the current repair iteration when OPEN is empty (which indicates that no path exists from $s_{\text{start}}$ to $s_{\text{goal}}$ or when the $f$-value of $s_{\text{goal}}$ is no larger than the smallest $f$-value of any cell in OPEN (which indicates that repeated ARA* found an $\epsilon$-suboptimal path from $s_{\text{start}}$ to $s_{\text{goal}}$, which can be traced in reverse by following the parents from $s_{\text{goal}}$ to $s_{\text{start}}$). If $\epsilon > 1$ and the user-provided time limit has not yet been reached, then repeated ARA* performs the following operations before the next repair iteration: Repeated ARA* decreases $\epsilon$ by a user-provided constant $\delta\epsilon$ (to find a path with a smaller error bound on its length), sets OPEN to the union of OPEN and INCONS, and sets CLOSED and INCONS to the empty sets. Repeated ARA* then starts the next repair iteration. It thus improves upon repeated weighted A*, which starts each repair iteration with OPEN containing only $s_{\text{start}}$ and thus does not reuse information from the previous repair iteration.

Figures 1(a-d) show a gridworld of size $7 \times 7$ that we use as running example throughout the paper. B2 is $s_{\text{start}}$ (marked “S”), and E6 is $s_{\text{goal}}$ (marked “G”). Cells in OPEN are marked “O” in the center, cells in CLOSED are marked “C” in the center, and cells in INCONS are marked “I” in the center. Arrows point from cells to their parents. We show the $g$-, $v$-, $h$- and $f$-values of each cell in its top-left, top-right, bottom-left and bottom-right corners, respectively. The $f$-values are shown only for cells in OPEN, CLOSED or INCONS. Ties among cells with the same $f$-values are broken in favor of cells with larger $g$-values because this typically results in smaller runtimes per repair iteration. Figures 1(a) and 1(b) show the cells before and after the first repair iteration, respectively. The first repair iteration uses $\epsilon = 10$, expands cells in the order B2, C2, D2, E2, F2, E4, F4, D4, D3, F3, C4, C5, C6 and D6, and returns a suboptimal path from B2 to E6 of length 9. Note that it expands D4 with $g(D4) = 6$ and later D3 with $g(D3) = 3$. At this point, it detects that $g(D3) + c(D3, D4) = 3 + 1 < 6 = g(D4)$, re-generates D4, and inserts it into INCONS. Figures 1(c) and 1(d) show the cells before and after the second repair iteration, respectively. The second repair iteration uses $\epsilon = 1$ and returns a shortest path of length 7.

**New Research: I-ARA***

I-ARA* operates in the same way as repeated ARA* but re-uses part of the search tree at the end of the last repair iteration of the previous I-ARA* search as the search tree in the beginning of the first repair iteration of the current I-ARA* search, namely the subtree that is rooted in $s_{\text{start}}$. Figure 3 shows the pseudocode of I-ARA*. The first I-ARA* search is an A* search. Thus, Figure 1 shows the first I-ARA* search. Figure 2 shows the second I-ARA* search after the hunter moved from B2 to C2 and the target moved from E6 to F6. The second I-ARA* search proceeds as follows, using the additional set $\text{DELETED}$:

**Step 1 (Making $s_{\text{start}}$, Locally Consistent)** $s_{\text{start}}$ never needs to be re-expanded since the search tree trivially contains an $\epsilon$-suboptimal path from $s_{\text{start}}$ to itself (namely, the empty path). To avoid unnecessary re-expansions of $s_{\text{start}}$ and potentially many of its descendants in the search tree, I-ARA* makes $s_{\text{start}}$ locally consistent if it is locally inconsistent, by setting its $g$-value to its $v$-value: I-ARA* then deletes $s_{\text{start}}$ from OPEN and INCONS. In our example, I-ARA* skips Step 1 since C2 is locally consistent already.

**Step 2 (Deleting Cells from the Previous Search Tree)** If $s_{\text{start}}$ is different from the previous cell of the hunter previous$\\_s_{\text{start}}$, I-ARA* sets the parent of $s_{\text{start}}$ to NULL and deletes all cells from the previous search tree that are not in the subtree rooted in $s_{\text{start}}$ by setting their $g$-values and $v$-values to infinity, setting their parents to NULL and deleting them from OPEN and INCONS. All cells deleted from the previous search tree are inserted into DELETED. In our example, I-ARA* deletes B2 and B3 and inserts them into $\text{DELETED}$.

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3$\text{DELETED}$ is the set of all cells that have been deleted from the search tree in the current repair iteration.
DELETED. All cells in DELETED are marked “D” in the center, see Figure 2(a).

**Step 3 (Completing OPEN)** OPEN is still incomplete since it is missing all cells in DELETED that are neighbors of at least one cell with a finite \( v \)-value and missing all cells in INCONS. Therefore, I-ARA* iterates through all cells in DELETED and performs the following check: If a cell \( s' \) in DELETED is a neighbor of at least one cell with a finite \( v \)-value, I-ARA* sets \( g(s') \) to \( v(s') + c(s', s) \) and \( parent(s') \) to \( s \), where \( s' \) is arg min \( \{ \text{Neighbor}(s') \mid v(s') + c(s', s') \} \), and inserts \( s' \) into OPEN. In our example, I-ARA* inserts \( B2 \) and \( B3 \) into OPEN. It then inserts all cells in INCONS into OPEN and sets CLOSED, INCONS and DELETED to the empty sets, see Figure 2(b).

**Step 4 (Updating Weight \( \epsilon \))** If the \( f \)-value of \( s_{goal} \) is not larger than the smallest \( f \)-value of all cells in OPEN, then the search tree already contains an \( \epsilon \)-suboptimal path from \( s_{start} \) to \( s_{goal} \). Thus, I-ARA* decreases the weight \( \epsilon \) by the user-provided constant \( \delta \), setting \( \epsilon \) to \( \max(1, \epsilon - \delta) \) to find a path with a smaller error bound on its length. Otherwise, I-ARA* has yet to find an \( \epsilon \)-suboptimal path. Thus, it sets weight \( \epsilon \) to the user-provided initial weight \( \epsilon_{max} \) to find a path with a large error bound on its length quickly. In our example, the second case holds and I-ARA* sets the weight to \( \epsilon_{max} = 10 \), see Figure 2(c).

**Step 5 (Starting Repair Iterations)** I-ARA* starts the next repair iteration of the current I-ARA* search. In our example, this repair iteration with weight 10 expands only \( E6 \), see Figure 2(d). In contrast, it would expand \( C2, C3, C4, C5, C6, D6 \) and \( E6 \) without re-using part of the search tree at the end of the last repair iteration of the previous I-ARA* search.

**Correctness of I-ARA**

Due to space constraints, we are only able to sketch the correctness proof of I-ARA*, that is, that ImprovePath() with weight \( \epsilon \) finds an \( \epsilon \)-suboptimal path from \( s_{start} \) to \( s_{goal} \) if one exists and reports that no path exists otherwise. We start with a useful lemma.

**Lemma 1.** For all cells \( s \in S \), \( g(s) \leq v(s) \).

**Proof.** Initially, \( g(s) = v(s) = \infty \). Whenever \( g(s) \) or \( v(s) \) changes, they are both set to infinity, \( v(s) \) is set to \( g(s) \), or \( g(s) \) is reduced. The lemma follows from these observations.

Corollary 15 in (Likhachev 2005) can be adapted to prove that \( g(s_{goal}) - g(s_{start}) \leq \epsilon \times c(s_{start}, s_{goal}) \) when ImprovePath() with weight \( \epsilon \) terminates if (A1) CLOSED is the empty set, (A2) OPEN is the set of all locally inconsistent cells, and (A3) for all cells \( s \in S \setminus \{ s_{start} \}, g(s) = v(parent(s)) + c(parent(s), s) = \min_{s' \in \text{Neighbor}(s)}(v(s') + c(s', s)) \) when ImprovePath() is called. 4 In the following,

4The version of ImprovePath() used in (Likhachev 2005) does not use INCONS and requires that \( g(s_{start}) = 0 \). The version of ImprovePath() used by I-ARA* uses INCONS only for bookkeeping and the proof in (Likhachev 2005) applies to it if one uses \( g(s) - g(s_{start}) \) instead of \( g(s) \) for all cells \( s \in S \).

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**Function ImprovePath()**

01. function ImprovePath()
02. while \( g(s_{goal}) + \epsilon \times h(s_{goal}, s_{goal}) > \min_{s \in \text{OPEN}(g(s) + \epsilon \times h(s, s_{goal}))} \)
03. move \( s \in \text{OPEN} \) with the smallest \( g(s) + \epsilon \times h(s, s_{goal}) \) from OPEN to CLOSED;
04. \( v(s) := g(s) \);
05. forall \( s' \in \text{Neighbor}(s) \)
06. if \( g(s') > v(s) + c(s', s) \)
07. \( g(s') := v(s) + c(s', s) \);
08. \( parent(s') := s \);
09. if \( s' \notin \text{CLOSED} \)
10. \( \text{if s' \notin OPEN AND s' \notin INCONS} \)
11. insert \( s' \) into OPEN;
12. else
13. move \( s' \) from CLOSED to INCONS;
14. return true;
15. repeat
16. return false if ImprovePath() = false;
17. return false if \( |\text{CLOSED}| = \infty \) OR the time limit has been reached;
18. \( \text{CLOSED} := \text{INCONS} := \emptyset \);
19. \( \epsilon := \max(1, \epsilon - \delta) \);
20. procedure Step1()
21. if \( g(s_{start}) \neq g(s_{start}) \)
22. \( v(s_{start}) := v(s_{start}) \);
23. delete \( s_{start} \) from INCONS if \( s_{start} \notin \text{INCONS} \);
24. delete \( s_{start} \) from OPEN if \( s_{start} \notin \text{OPEN} \);
25. procedure Step2()
26. if \( g(s) \neq g(s_{goal}) \)
27. \( g(s_{goal}) := v(s_{goal}) \);
28. delete \( s_{goal} \) from CLOSED if \( s_{goal} \notin \text{CLOSED} \);
29. delete \( s_{goal} \) from OPEN if \( s_{goal} \notin \text{OPEN} \);
30. procedure Step3()
31. forall \( s \in \text{DELETED} \)
32. if \( \text{parent}(s) = \text{NULL} \)
33. delete \( s \) from \text{INCONS} if \( s \notin \text{INCONS} \);
34. delete \( s \) from OPEN if \( s \notin \text{OPEN} \);
35. insert \( s \) into \text{DELETED};
36. procedure Step4()
37. forall \( s \in \text{DELETED} \)
38. \( \text{if } g(s_{goal}) + \epsilon \times h(s_{goal}, s_{goal}) > \min_{s \in \text{OPEN}}(g(s) + \epsilon \times h(s, s_{goal})) \)
39. \( g(s) := g(s_{goal}) + \epsilon \times h(s_{goal}, s_{goal}) \);
40. \( \epsilon := \epsilon_{max} \);
41. return true;
42. else
43. return false;
44. procedure Main()
45. forall \( s \in S \)
46. \( v(s) := g(s) = \infty \);
47. \( \text{parent}(s) = \text{NULL} \);
48. \( \epsilon := \epsilon_{max} \);
49. \( s_{start} \) := the current cell of the hunter;
50. \( s_{goal} \) := the current cell of the target;
51. \( \text{OPEN} := \text{CLOSED} := \text{DELETED} := \emptyset \);
52. \( g(s_{start}) := 0 \);
53. insert \( s_{start} \) into OPEN;
54. while \( s_{start} \neq s_{goal} \)
55. return false if ComputePath() = false; /* Step 5 */
56. \( \text{previous}(s_{start}) := s_{start} \);
57. identify a path from \( s_{start} \) to \( s_{goal} \) using the parents;
58. if the path has not been caught yet AND is still on the path from \( s_{start} \) to \( s_{goal} \)
59. the hunter follows the path from \( s_{start} \) to \( s_{goal} \);
60. \( s_{start} := \text{the current cell of the hunter} \);
61. \( s_{goal} := \text{the current cell of the target} \);
62. \( \text{Step1()} \); /* Step 1 */
63. \( \text{Step2()} \); /* Step 2 */
64. \( \text{Step3()} \); /* Step 3 */
65. \( \text{Step4()} \); /* Step 4 */
66. return true;

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**Figure 3: I-ARA**
we first prove the premise by proving A1, A2 and A3 separately. We then use the conclusion for the correctness proof of I-ARA$^{*}$.

We now prove that A1 holds whenever ImprovePath() is called. This is easy to see since both CLOSED and INCONS are always set to the empty set before and between calls to ImprovePath() (Lines 23, 48 and 61).

We now prove that A2 holds whenever ImprovePath() is called. We start with a useful lemma.

**Lemma 2.** OPEN $\cup$ INCONS is the set of all locally inconsistent cells when ImprovePath() terminates if A2 holds when ImprovePath() is called.

**Proof.** We prove the lemma by induction on the number of times ImprovePath() executes Line 2: The conclusion of the lemma holds for the first execution of Line 2 since A2 holds due to the premise of the lemma and CLOSED and INCONS are the empty set. The conclusion of the lemma also holds for each subsequent execution of Line 2: ImprovePath() sets the $v$-value of a cell to its $g$-value on Line 4 and thus makes it locally consistent, and it might reduce the $g$-value of a cell on Line 7, thus making it locally inconsistent according to Lemma 1. Whenever a cell is made locally consistent, it is deleted from OPEN $\cup$ INCONS (Line 3). Whenever a cell is made locally inconsistent, it is inserted into OPEN $\cup$ INCONS (Lines 11 and 13). These are the only ways how cells can be inserted into or deleted from OPEN $\cup$ INCONS.

Next, we prove that A2 holds whenever ImprovePath() is called if A2 holds whenever ComputePath() is called. We prove the statement by induction on the number of times ComputePath() calls ImprovePath(). A2 holds when ComputePath() calls ImprovePath() for the first time since A2 holds whenever ComputePath() is called. A2 also holds when ComputePath() calls ImprovePath() each subsequent time: OPEN $\cup$ INCONS is the set of all locally inconsistent cells when ImprovePath() terminates according to Lemma 2. ComputePath() then restores A2 since it inserts all cells in INCONS into OPEN (Line 22).

Finally, we prove that A2 holds whenever ComputePath() is called. We prove the statement by induction on the number of times Main() calls ComputePath(). A2 holds when Main() calls ComputePath() for the first time since OPEN contains the only locally inconsistent cell, namely $s_{\text{start}}$. A2 also holds when Main() calls ComputePath() each subsequent time: OPEN $\cup$ INCONS is the set of all locally inconsistent cells when ComputePath() terminates since it is the set of all locally inconsistent cells when ImprovePath() terminates according to Lemma 2. Main() then restores A2 since it inserts all cells in INCONS into OPEN (Line 47).

We have proven that A2 holds whenever ImprovePath() is called if A2 holds whenever ComputePath() is called. We have also proven that A2 holds whenever ComputePath() is called. Consequently, we have proven that A2 holds whenever ImprovePath() is called.

We now prove that A3 holds whenever ImprovePath() is called. We first prove that A3 holds whenever ImprovePath() terminates if A3 holds whenever ImprovePath() is called. We prove the statement by induction on the number of times ImprovePath() executes Line 2, taking into account Lines 4-8. We then use this statement to prove that A3 holds whenever ImprovePath() is called if A3 holds whenever ComputePath() is called. We prove this statement, in turn, by induction on the number of times ComputePath() calls ImprovePath(), taking into account that ImprovePath() but not the rest of ComputePath() can change the $g$- and $v$-values. Finally, we prove that A3 holds whenever ComputePath() is called. We prove the statement by induction on the number of times Main() calls ComputePath(). A3 holds when Main() calls ComputePath() for the first time due to the initialization of the $g$-values, parents, and $v$-values of all cells (Lines 56-57 and 62). A3 also holds when Main() calls ComputePath() each subsequent time: A3 holds whenever ComputePath() terminates since it holds whenever ImprovePath() terminates. We prove that A3 also holds after Main() executes Line 76 if it holds before Main() executes Line 66: If the hunter does not move and its current cell $s_{\text{start}}$ is still its previous cell $s_{\text{prev}}$, then the $v$-values, parents and $v$-values of no cell changes when Main() executes Lines 66-76 and A3 continues to hold. If the hunter moves, we partition all cells into three sets: $S_{\text{new}}$ is the set of all cells in the subtree of the search tree rooted in $s_{\text{start}}$, $S_{\text{old}}$ is the set of all cells in the search tree (rooted in previous $s_{\text{start}}$) but not in $S_{\text{new}}$, and $S_{\text{rest}}$ is the set of all remaining cells. Consider any cell $s \in S \setminus \{s_{\text{start}}\}$.

- If $s \in S_{\text{new}}$, then $\text{parent}(s) \in S_{\text{new}}$ before Main() executes Line 66 since $S_{\text{new}}$ is the set of all cells in a subtree of the search tree. Neither the $g$-value of $s$, the parent of $s$, nor the $v$-value of $\text{parent}(s)$ changes when Main() executes Lines 66-76, $\min_{s' \in \text{Neighbor}(s)}(v(s') + c(s', s))$ does not decrease since the $v$-value of no cell decreases. Consequently, $s$ does not violate A3 after Main() executes Line 76.

- If $s \in S_{\text{rest}}$, then the $g$-value of $s$ is infinity before Main() executes Line 66 since $S_{\text{rest}}$ is the set of cells that are not in the search tree and whose $g$-values are thus infinity. For all $s' \in \text{Neighbor}(s)$, the $v$-value of $s'$ is also infinity according to A3. Neither the $g$-value nor parent of $s$ changes when Main() executes Lines 66-76, $\min_{s' \in \text{Neighbor}(s)}(v(s') + c(s', s))$ does not decrease since the $v$-value of no cell decreases. Consequently, $s$ does not violate A3 after Main() executes Line 76.

- If $s \in S_{\text{old}}$, then the $g$- and $v$-value of $s$ set to infinity on Line 34. $\text{DELETED}$ is the empty set (Lines 48 and 61) before the cells in $S_{\text{old}}$ are inserted into it (Lines 33 and 38) and thus $S_{\text{old}}$ is the set of all cells in $\text{DELETED}$. The $g$-value of $s$ is then set to $\min_{s' \in \text{Neighbor}(s)}(v(s') + c(s', s))$,
Lemma 1. \( \text{ImprovePath}(i) \) always expands a cell in OPEN and \( \text{ImprovePath}() \) eventually terminates. We have shown that \( \text{ImprovePath}(i) \) with weight \( \epsilon \) terminates if \( \text{A1}, \text{A2}, \text{and A3} \) hold when \text{ImprovePath}(i) is called.

We have also proven that \( \text{A1}, \text{A2}, \text{and A3} \) hold when \text{ImprovePath}(i) is called.

Prove. We have shown that \( \text{A1}, \text{A2}, \text{and A3} \) hold whenever \text{ImprovePath}(i) with weight \( \epsilon \) terminates. We now use this result for the correctness proof of I-ARA*.

**Theorem 1.** \( \text{ImprovePath}(i) \) returns true if a path from \( s_{\text{start}} \) to \( s_{\text{goal}} \) exists and false otherwise.

**Proof.** \text{ImprovePath}(i) always expands a cell in OPEN and then does not reinsert it into OPEN. Since \( \epsilon \) is finite, \text{ImprovePath}(i) eventually terminates. We have shown that \( g(s_{\text{goal}}) - g(s_{\text{start}}) \leq \epsilon \times c(s_{\text{start}}, s_{\text{goal}}) \) when \text{ImprovePath}(i) with weight \( \epsilon \) terminates. If a path from \( s_{\text{start}} \) to \( s_{\text{goal}} \) exists, then \( g(s_{\text{goal}}) - g(s_{\text{start}}) \) is finite, which means that \( g(s_{\text{goal}}) \) is also finite (since \( g(s_{\text{start}}) \) is finite) and \text{ImprovePath}(i) returns true (Lines 14 and 17). If no such path exists, then \( g(s_{\text{goal}}) \) is infinite (which can be proven by contradiction using Lemma 1 and A3) and \text{ImprovePath}(i) returns false (Lines 14-15).

**Theorem 2.** When \text{ImprovePath}(i) with weight \( \epsilon \) returns true, an \( \epsilon \)-suboptimal path from \( s_{\text{start}} \) to \( s_{\text{goal}} \) of length at most \( g(s_{\text{goal}}) - g(s_{\text{start}}) \) can be traced in reverse by following the parents from \( s_{\text{goal}} \) to \( s_{\text{start}} \).

**Proof.** We have shown that \( g(s_{\text{goal}}) - g(s_{\text{start}}) \leq \epsilon \times c(s_{\text{start}}, s_{\text{goal}}) \) and A3 holds whenever \text{ImprovePath}(i) with weight \( \epsilon \) terminates. A3 and Lemma 1 ensure that the \( g \)-values of the cells are strictly monotonically decreasing when following the parents from \( s_{\text{goal}} \). One eventually reaches \( s_{\text{start}} \) since all cells with finite \( g \)-values, except for \( s_{\text{start}} \), have parents. Now consider the resulting path \((s_0 = s_{\text{start}}, \ldots, s_k = s_{\text{goal}})\). We prove that the path \((s_0 = s_{\text{start}}, \ldots, s_k = s_{\text{goal}})\) is of length at most \( g(s_k) - g(s_{\text{start}}) \) by induction on \( i \): The path \((s_0 = s_{\text{start}}, \ldots, s_0 = s_{\text{start}})\) is trivially of length at most \( g(s_0) - g(s_{\text{start}}) = 0 \). Now assume that the path \((s_0 = s_{\text{start}}, \ldots, s_i = s_{\text{start}})\) for \( i > 0 \) is of length at most \( g(s_i) - g(s_{\text{start}}) \) and thus \( g(s_{i+1}) - g(s_i) \) according to A3 and Lemma 1 since \( \text{parent}(s_{i+1}) = s_i \). The length of path \((s_0 = s_{\text{start}}, \ldots, s_{i+1})\) is equal to the sum of the length of path \((s_0 = s_{\text{start}}, \ldots, s_i)\), which is at most \( g(s_i) - g(s_{\text{start}}) \), and \( c(s_i, s_{i+1}) \), which is at most \( g(s_{i+1}) - g(s_i) \).

**Experimental Results**

Moving-target search algorithms in known terrain can be classified into off-line and on-line algorithms. Off-line moving-target search algorithms, such as Reverse Minimax A* (Moldenhauer and Sturtevant 2009), determine the optimal strategy for the hunter once by taking into account all strategies for the target, which tends to make their searches too slow on larger maps. On-line search algorithms react to the actual moves of the target and thus need to search repeatedly. Some on-line search algorithms use the strategy for the hunter to find a path from \( s_{\text{start}} \) to \( s_{\text{goal}} \) and move along it. If the target moves off the path, then they repeat the process. Real-time moving-target search algorithms, such as MTS (Ishida and Korf 1991), find a prefix of a path in constant time, which tends to result in a large number of moves for the hunter until it catches the target or needs a large amount of memory (Bulitko et al. 2007b). We therefore study moving-target search algorithms that find a complete path, for example, via repeated A* or weighted A* searches. Incremental search has been used to speed up both kinds of moving-target search algorithms. For example, USP (Edelkamp 1998) is an incremental version of a version of MTS (Sasaki, Chimura, and Tokoro 1995), and G-FRA* (Sun, Yeoh, and Koenig 2010) is an incremental version of repeated A*. G-FRA* finds shortest paths for the hunter and is the currently fastest such incremental moving-target search algorithm on known graphs. FRA* optimizes it for gridworlds (Sun, Yeoh, and Koenig 2009).

We therefore compare I-ARA* experimentally to repeated A*, G-FRA*, FRA* and repeated ARA* (Likhachev, Gordon, and Thrun 2003). It is important to realize that experimental results, such as the runtimes of the search algorithms, depend on a variety of factors, including implementation details (such as the data structures, tie-breaking strategies, and coding tricks used) and experimental setups (such as whether the gridworlds are four-neighbor or eight-neighbor gridworlds). We do not know of any better method for evaluating search algorithms than to implement them as best as possible, publish their runtimes, and let other researchers experiment with their own and thus potentially different implementations and experimental setups. For fairness, we use comparable implementations. For example, all search algorithms use binary heaps as priority queues and replan when the target moves off the path. We perform our experiments in (1) four-neighbor gridworlds of size \( 1,000 \times 1,000 \) with 25 percent randomly blocked cells and (2) a four-neighbor video game map of size 676 \times 676 adapted from Warcraft III (courtesy of Nathan Sturtevant). We average over 100 test cases (corresponding to 100 different gridworlds but the same video game map) with randomly selected unblocked cells for the hunter and target for both kinds of maps, with the restriction that a path exists between both cells. The target always follows a shortest path from its current cell to a randomly selected unblocked cell and repeats the process once it reaches that cell. It skips every tenth move, which allows the hunter to catch it in all cases.

**Experiment 1**

We first let I-ARA* and repeated ARA* perform only one repair iteration per search with a user-provided weight, which we vary from 1.0 to 5.0. Thus, repeated ARA* reduces to repeated weighted A*. Table 1 reports one measure for the solution quality of the search algorithms, namely the
number of moves of the hunter until it catches the target. The ratio of this number and the number of moves of the hunter of repeated A* is shown in square brackets. The table reports three measures for the efficiency of the search algorithms, namely the number of expanded cells per search and two runtimes in microseconds on an Intel Xeon 3.20 GHz PC with 2 GB of RAM, namely the average runtime per search over all searches until the hunter catches the target and the runtime of the number of expanded cells per search, the average runtime per search and the largest runtime of any search until the hunter catches the target. The trends for the number of expanded cells per search, the average runtime per search and the largest runtime of any search are similar, and we thus refer only to the runtime per search. The standard deviation of the mean for the number of expanded cells per search is shown in parentheses. We notice the following relationships:

- The numbers of moves of all search algorithms tend to be similar when I-ARA* and repeated ARA* use weight one since they all find the same paths (modulo tie breaking).
- The runtime per search of repeated A* tends to be smaller than the one of repeated ARA* with weight one since they operate in the same way and thus tend to have the same number of expanded cells per search but repeated ARA* needs extra bookkeeping operations. The runtime per search of I-ARA* with weight one tends to be smaller than the one of repeated A* since it uses incremental search. The runtime per search of G-FRA* tends to be smaller than the one of I-ARA* with weight one since they operate in the same way and thus tend to have the same number of expanded cells per search but I-ARA* needs extra bookkeeping operations. The runtime per search of FAA* tends to be smaller than the one of G-FRA* since it is optimized for gridworlds.
- The numbers of moves of I-ARA* tends to be smaller than the one of repeated ARA* with the same weight and both increase slowly as the weight increases since they trade-off solution quality and efficiency.

Table 1: User-Provided Error Bound $\epsilon$ on the Length of the Path of each Search

<table>
<thead>
<tr>
<th>Gridworlds</th>
<th>Game Map</th>
</tr>
</thead>
<tbody>
<tr>
<td>moves per test case</td>
<td>expansions per test case</td>
</tr>
<tr>
<td>$t = 1000\mu s$</td>
<td>$t = 2000\mu s$</td>
</tr>
<tr>
<td>I-ARA*</td>
<td>792 (1.00)</td>
</tr>
<tr>
<td>ARA*</td>
<td>792 (1.00)</td>
</tr>
<tr>
<td>FRA*</td>
<td>792 (1.00)</td>
</tr>
</tbody>
</table>

Table 2: User-Provided Time Limit $t$ for each Search

<table>
<thead>
<tr>
<th>Gridworlds</th>
<th>Game Map</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>FRA*</td>
<td>792 (1.00)</td>
</tr>
</tbody>
</table>
The runtime per search of I-ARA* tends to be smaller than the one of repeated ARA* with the same weight since it uses incremental search to re-use information from the last repair iteration of the previous search to speed up the first repair iteration of the current search. The runtimes per search of I-ARA* and repeated ARA* with the same weight tend to decrease as the weight increases. The runtime per search of I-ARA* tends to be smaller than the ones of all other tested search algorithms for sufficiently large weights.

The advantage of I-ARA* over G-FRA* and repeated ARA* searches is due to the following reasons:

- The average runtime per search of G-FRA* tends to be smaller than the one of repeated A* but its largest runtime of any search tends to be only slightly smaller. Both the average runtime per search and the largest runtime of any search of I-ARA* with weights larger than one tend to be smaller than the ones of G-FRA* but the largest runtime of any search tends to be much smaller. The reason for this is that the first search is often with the largest runtime. The first G-FRA* search is an A* search (and the largest runtime of any search of G-FRA* thus tends to be similar to the one of repeated A*), while the first I-ARA* search is a faster weighted A* search.

- Both the average runtime per search and the largest runtime of any search of I-ARA* tend to be smaller than the ones of repeated ARA* with the same weight but the average runtime per search tends to be much smaller. The reason for this is that the first search is often the one with the largest runtime. The first searches of I-ARA* and repeated ARA* operate in the same way (and the largest runtime of any search of I-ARA* thus tends to be similar to the one of repeated ARA*), while I-ARA* uses incremental search to speed up the subsequent searches.

### Experiment 2

We now impose a time limit per search, which we vary from 100 to 3,000 µs. Since all search algorithms have to find a path from $s_{start}$ to $s_{goal}$, we let FRA* complete its searches independent of the time limit and repeated ARA* and I-ARA* complete their first repair iterations of all search trees independent of the time limit. We use $\epsilon_{max} = 2.0$ and $\delta_t = 0.1$ for repeated ARA* and I-ARA*. Table 2 reports, in addition to some of the measures from Table 1, the largest runtime of any repair iteration per search, the number of repair iterations per search, and the percentage of searches that exceed the time limit. The percentage of searches different from the first search that exceed the time limit is shown in curly braces since the first search is performed before the hunter starts to move and thus might not be subject to the time limit. We notice the following relationships:

- The number of repair iterations per search of I-ARA* tends to be larger than the one of repeated ARA* with the same time limit, as already suggested by the average runtime per search in Experiment 1. Consequently, I-ARA* tends to be able to provide smaller error bounds on the lengths of the paths found than repeated ARA* and its number of moves tends to be smaller since it tends to be able to decrease the weight more. The number of moves of I-ARA* tends to be similar to the one of FRA* for sufficiently large time limits.

- The percentage of searches of I-ARA* that exceed the time limit tends to be smaller than the one of repeated ARA* since the largest runtime of any repair iteration per search of I-ARA* tends to be smaller than the one of repeated ARA*, as already suggested by the largest runtime of any search in Experiment 1. (The largest runtimes of any repair iteration tend not to depend on the time limit since the first repair iteration per search is often one with the largest runtime.) Large time limits in gridworlds are an exception: Large time limits allow I-ARA* to perform searches with smaller weights, which result in larger search trees, whose preprocessing times can sometimes exceed the time limit.

- The percentage of searches of I-ARA* that exceed the time limit tends to be smaller than the one of FRA*, as already suggested by the largest runtime of any search in Experiment 1. Small time limits are an exception: The time needed to preprocess the search trees is often longer for I-ARA* than FRA* (and can be longer than the time limit for small time limits) since it needs extra bookkeeping operations and it is not optimized for gridworlds.

So far, all search algorithms replanned when the target moved off the path (marked ‘Off the Path’). However, the hunter might have the time to perform a search before every move (marked ‘Every Move’). The number of moves of repeated ARA* remains about the same, but the number of moves of I-ARA* decreases slightly since it uses incremental search, see Table 3.

### Conclusions

We developed Incremental ARA* (I-ARA*), the first incremental anytime search algorithm for moving-target search in known terrain. We provided an error bound on the lengths of the paths found by I-ARA* and showed experimentally in known four-neighbor gridworlds that I-ARA* can be used with smaller time limits between moves of the hunter than competing state-of-the-art moving-target search algorithms. The hunter tends to make more moves with I-ARA* than repeated A*, G-FRA* or FRA*, which find shortest paths for the hunter, but fewer moves with I-ARA* than repeated ARA*, which finds suboptimal paths for the hunter like I-ARA*. Also, the error bounds on the lengths of the paths of the hunter tend to be smaller with I-ARA* than repeated ARA*.
References


