First-Order Logic– Solution

1) Translate the following English sentences to first-order logic using the following predicates: Owns(x, y), Dog(x), Cat(x), Cute(x), and Scary(x). For example, Owns(x, y) means that object x owns object y:

(a) Joe has a cute dog.
Answer: $\exists x$ (Owns(Joe, x) $\wedge$ Dog(x) $\wedge$ Cute(x))

(b) All of Joe's dogs are cute.
Answer: $\forall x$ ((Owns(Joe, x) $\wedge$ Dog(x)) $\Rightarrow$ Cute(x))

(c) Unless Joe owns a dog, he is scary.
Answer: $\neg$($\exists x$ (Owns(Joe, x) $\wedge$ Dog(x))) $\Rightarrow$ Scary(Joe)

(d) Either Joe has at least one cat and at least one dog or he is scary (but not both at the same time).
Answer: ($\exists x$ (Owns(Joe, x) $\wedge$ Dog(x))) $\wedge$($\exists y$ (Owns(Joe, y) $\wedge$ Cat(y))) $\Leftrightarrow$ $\neg$ Scary(Joe).

(e) Not all dogs are both scary and cute.
Answer: $\exists x$ (Dog(x) $\wedge\neg$ (Scary(x) $\wedge$ Cute(x)))

2) Translate the following sentences in first-order logic to English. Apple(x) means that object x is an apple, Red(x) means that object s is red, Loves(x, y) means that person x loves person y:

(a) $\forall x$ (Apple(x) $\Rightarrow$Red(x))
Answer: All apples are red.

(b) $\forall x$ $\exists y$ Loves(x, y)
Answer: Every person has some person he loves.

(c) $\exists y$ $\forall x$ Loves(x, y)
Answer: There is a single person whom everybody loves.

3) Specify what a grandmother is, using the predicates IsGrandMotherOf, IsMotherOf and IsFatherOf. IsGrandMotherOf(x, y) means that person x is the grandmother of person y, IsMotherOf(x, y) means that person x is the mother of person y, and IsFatherOf(x, y) means that person x is the father of person y. Define additional predicates if needed.

Answer:
$\forall x, y$ (IsGrandMotherOf(x, y) $\Leftrightarrow$
$\exists z$ (IsMotherOf(x, z) $\wedge$ (IsMotherOf(z, y) $\vee$ IsFatherOf(z, y))))

4) For each of the following sentences in first-order logic, specify whether it is valid, satisfiable, and/or unsatisfiable:

(a) $P(A) \Rightarrow \forall x P(x)$
Answer: Satisfiable but not valid.

(b) $P(A) \Rightarrow \forall x \neg P(x)$
Answer: Satisfiable but not valid.

(c) $P(A) \Rightarrow \exists x P(x)$
Answer: Valid.

(d) $P(A) \Rightarrow \exists x \neg P(x)$
Answer: Satisfiable but not valid.

5) Solve Problem 9.23 on page 365 of our textbook.

Answer:

(a) Horses are animals:
$\forall x\ (\text{Horse}(x) \Rightarrow \text{Animal}(x))$

The head of a horse is the head of an animal:
$\forall h\ ((\exists y\ (\text{HeadOf}(h, y) \land \text{Horse}(y))) \Rightarrow (\exists z\ (\text{HeadOf}(h, z) \land \text{Animal}(z))))$

(b) Horses are animals (CNF):
$\forall x\ (\neg\text{Horse}(x) \lor \text{Animal}(x))$
$\neg\text{Horse}(x) \lor \text{Animal}(x)$

The head of a horse is the head of an animal (CNF after negation):
$\neg\forall h\ ((\exists y\ (\text{HeadOf}(h, y) \land \text{Horse}(y))) \Rightarrow (\exists z\ (\text{HeadOf}(h, z) \land \text{Animal}(z))))$
$\exists h\ \neg((\exists y\ (\text{HeadOf}(h, y) \land \text{Horse}(y))) \Rightarrow (\exists z\ (\text{HeadOf}(h, z) \land \text{Animal}(z))))$
$\exists h\ \neg(\neg(\exists y\ (\text{HeadOf}(h, y) \land \text{Horse}(y))) \lor (\exists z\ (\text{HeadOf}(h, z) \land \text{Animal}(z))))$
$\exists h\ ((\exists y\ (\text{HeadOf}(h, y) \land \text{Horse}(y))) \land \neg(\exists z\ (\text{HeadOf}(h, z) \land \text{Animal}(z))))$
$\exists h\ ((\exists y\ (\text{HeadOf}(h, y) \land \text{Horse}(y))) \land (\forall z\ (\neg\text{HeadOf}(h, z) \lor \neg\text{Animal}(z))))$
$(\text{HeadOf}(H, Y) \land \text{Horse}(Y)) \land (\neg\text{HeadOf}(H, z) \lor \neg\text{Animal}(z))$
$\text{HeadOf}(H, Y)$, $\text{Horse}(Y)$, $\neg\text{HeadOf}(H, z) \lor \neg\text{Animal}(z)$

(c) We start with the four clauses we have derived in (b):
(1) $\neg\text{Horse}(x) \lor \text{Animal}(x)$
(2) $\text{HeadOf}(H, Y)$
(3) $\text{Horse}(Y)$
(4) $\neg\text{HeadOf}(H, z) \lor \neg\text{Animal}(z)$

(5) (from 2 and 4, $z = Y$) $\neg\text{Animal}(Y)$
(6) (from 1 and 5, $x = Y$) $\neg\text{Horse}(Y)$
(7) (from 3 and 6) $\perp$